Robust Control of Underactuated Wheeled Mobile Manipulators using GPI Disturbance Observers

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Abstract This article describes the design of a linear observer-linear controller-based robust output feedback scheme for output reference trajectory tracking tasks in the case of nonlinear, multivariable, nonholonomic underactuated mobile manipulators. The proposed linear feedback scheme is based on the use of a classical linear feedback controller and suitably extended, high gain, linear Generalized Proportional Integral (GPI) observers, thus aiding the linear feedback controllers to provide an accurate simultaneous estimation of each flat output associated phase variables and of the exogenous and perturbation inputs. This information is used in the proposed feedback controller in a) approximate, yet close, cancelations, as lumped unstructured time-varying terms, of the influence of the highly coupled nonlinearities and b) the devising of proper linear output feedback control laws based on the approximate estimates of the string of phase variables associated with the flat outputs simultaneously provided by the disturbance observers. Simulations reveal the effectiveness of the proposed approach.

1 Introduction.

Mobile manipulator systems consist of a mobile platform and a manipulator arm mounted on the platform. In a number of operation tasks, they combine the advantages of mobile platforms and those of the conventional manipulators while reducing their drawbacks. In particular, mobile manipulators increase the workspace limitations of fixed-based manipulators, they posses higher kinematics redundancy and are able to simultaneously operate and move around. A considerable amount of research has been carried out in the past as a result of having recognized these advantages, and many applications can be found in the areas of: construction, forestry, planetary exploration, medical treatment, electronic assembling, cooperative payload transport and the military among others [1]-[6]. However, the mobile manipulator has high dynamic coupling between the mobile base and the manipulator arm while being subject to nonholonomic constraints, arising from wheel kinematics and natural limits on the motion capabilities of the platform. As Brockett's theorem states [7], the asymptotic stabilization of a nonholonomic system at a specified fixed configuration cannot be obtained by using smooth (or even continuous) pure state feedback. Due to Brockett's condition, the study of effective control methods which exploit the capabilities for nonholonomic mobile manipulators have received considerable attention from the scientific community. Lim and Seraji [8] proposed a control law for mobile manipulators and solved the redundant equations by using geometry based control scheme and weighted pseudo inverses. Bayle *et al.* [9][10] solved the kinematic control of mobile manipulators, and avoided singularities and maximized the arm manipulability by using a pseudo-inversion scheme to coordinate the evolution of the mobile platform and the robot arm. De Luca *et al.* [11] studied the kinematic control problem for nonholonomic mobile manipulators in the presence of steering wheels. Liu and Goldenberg [12] used robust damping control (RDC) for the motion control of mobile manipulators with kinematics constraints and in the presence of unknown bounded disturbances. Zhang *et al.* [13] included the use of the Least Squares Support Vector Machine to achieve the control of an n-DOF mobile manipulator mounted on a two-wheeled mobile platform. Other approaches with which to control mobile robotic manipulators include a fuzzy indirect adaptive sliding controller (see Medhaffar and Derbel [14]), neural networks (see Lee et al. [15]) or nonlinear backstepping methods (see Acar and Murakami [16]). A fully-actuated manipulator can perform any joint trajectory. However, when some of the actuators in the open chain manipulator are removed, the robot becomes underactuated and the properties of controllability and feedback linearizability may be lost for a given mass distribution. Interestingly enough, if the system is found to be *differentially flat*, (see Fliess *et al.* [17] for the original introduction for the flatness concept and the books of Sira-Ramírez and Agrawal [18] and Lévine [19] for interesting real life examples), it can still perform efficient trajectory tracking, including rest to rest maneuvers involving point to point displacements despite having fewer actuators [17]. In general, not all nonlinear dynamic systems satisfy the conditions of differential flatness. However, it is possible to achieve this property by altering the inertia distribution through counterbalancing as in [20]. Agrawal and his co-workers have developed interesting studies on the planning and control of mobile manipulators based on the *differential flatness* property which have been helpful in the preparation of this article. In particular, they demonstrated its control integration in a experimental mobile manipulator [21] and developed studies concerning the planning and control of these manipulators based on the differential flatness property [22]-[25].

In this work, we propose a fundamentally linear, global, approach for the robust output feedback controller design task for a class of nonholonomic wheeled mobile manipulator (WMM). We develop a robust GPI observer-based, linear output feedback controller for the trajectory tracking problem of autonomous underactuated mobile manipulators subject to nonholonomic constraints. The linear observer-based controller design approach, presented here, is most suitable for the ubiquitous class of differentially flat systems. The proposed control approach, denominated as Generalized Proportional Integral (GPI) observer-based control, rests on using highly simplified models on the input-to-flat output models derived from the flatness property. In this simplification, only the order of integration of the subsystems and the control inputs, along with their associated matrix gains are retained in full detail. All the additive nonlinearities, including the state couplings and complexities, are regarded as, unstructured, time-varying signals that need to be estimated on-line, and canceled, at the controller specification within an Active Disturbance Rejection Control Scheme. After input gain matrix cancelation, the resulting system consists of pure integration (linear) perturbed systems with time-varying additive disturbances. A set of linear extended observers, here denominated as GPI observers, are subsequently produced which internally model the state dependent additive nonlinearities as time-polynomials of reasonably low orders. The observers' state estimation errors are shown to satisfy a set of decoupled, perturbed, linear differential equations with assignable constant coefficients. Under the assumption that the exogenous time-varying perturbation inputs are uniformly absolutely bounded, the designed observers estimate each individual flat output's associated string of phase variables as well as the time-varying perturbation, or disturbance input components. The state and perturbation estimation relies on a high gain observer design. The flatness property of the mobile manipulator system allows a meaningful input-to-highest derivative of flat outputs relation to be obtained. The proposed linear feedback scheme is based on the use of a classical linear feedback controller and a suitably extended high gain linear observer; thus aiding the linear feedback controller in two important tasks: (1) accurate estimation of the input-output system model nonlinearities; and (2) accurate estimation of the unmeasured phase variables associated with each of the linearizing output variables. These two key pieces of information are used in the proposed feedback controller to: (a) cancel, as a lumped unstructured time-varying term, the influence of the nonlinearities and (b) devise a proper linear output feedback based on the approximate estimates of the flat output associated phase variables.

The paper is organized as follows: Section 2 presents the dynamics model of the wheeled mobile manipulator, and its flatness property is demonstrated. This section also proposes a simplified model of the system and formulates the problem to be solved. Section 3 introduces some generalities as regards state-dependent disturbance estimation-disturbance elimination linear output feedback strategy. The results obtained are applied to the stabilization and trajectory tracking problem in a nonholonomic two-wheeled differentially driven mobile manipulator. In this section, the proposed controller is also developed. Section 4 includes numerical simulations illustrating the performance of the proposed approach under large initial errors and

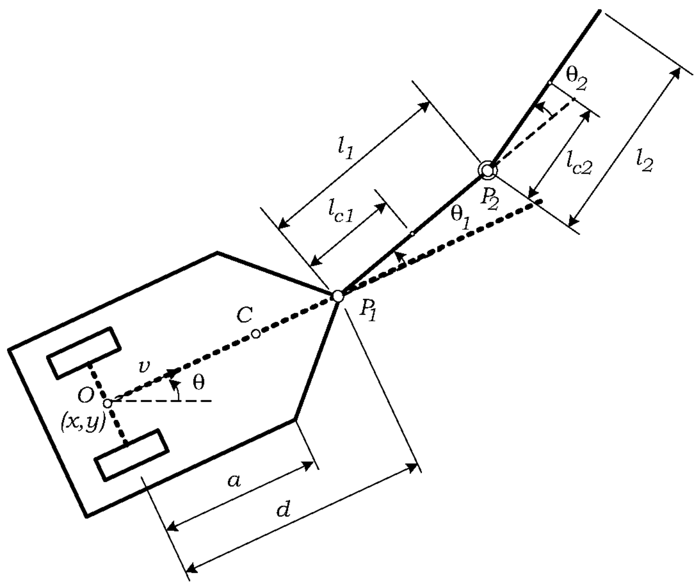

Fig. 1 General Scheme of a two-wheeled mobile base with a two-link planar manipulator arm

considerable parametric uncertainties in the model. Finally, Section 5 is devoted to the conclusions and suggestions for further work.

2 THE UNDER-ACTUATED MOBILE MANIPULATOR AND PROBLEM FORMULATION

This section is divided into several parts: It starts showing the complete dynamic model of the under-actuated mobile manipulator. The differential flatness property is then introduced for the mobile manipulator considered in this work. Next, a simplified model based on flat output dynamics is presented and finally, the formulation of the problem to be solved is stated.

2.1 Dynamic Model of the Underactuated Mobile Manipulator

The equations of motion of the mobile base (two active mobile wheels and one Swedish wheel) are modeled as a mechanical system with nonholonomic constraints of the form

$$\mathbf{M}_A(\mathbf{q}_A)\ddot{\mathbf{q}}_A + \mathbf{V}_A(\mathbf{q}_A, \dot{\mathbf{q}}_A) = \mathbf{E}_A(\mathbf{q}_A)\boldsymbol{\tau}_A - \mathbf{C}_A^T(\mathbf{q}_A)\boldsymbol{\lambda} - \mathbf{R} \tag{1}$$

$$\mathbf{C}_A(\mathbf{q}_A)\dot{\mathbf{q}}_A = \mathbf{0} \tag{2}$$

where $\mathbf{q}_A = [x, y, \theta]^T$ is a vector of generalized coordinates which describes the position and orientation of the mobile base, $\mathbf{M}_A$ is a symmetric positive-definite inertia matrix, $\mathbf{V}_A$ denotes the centripetal and Coriolis vector, $\mathbf{E}_A\boldsymbol{\tau}_A$ is the generalized force vector along $\mathbf{q}_A$ resulting from the wheel torques $\boldsymbol{\tau}_A$, $\mathbf{C}_A$ is a full-row rank matrix quantifying the nonholonomic constraints [26], $\boldsymbol{\lambda}$ is a vector of Lagrange multipliers which represent constraint forces, and $\mathbf{R}$ is the generalized force vector resulting from the interaction forces and torques applied by the manipulator arm on the mobile base at the point P_1 (see Figure 1). From [27][28] the matrix $\mathbf{S}_A(\mathbf{q}_A)$ is a full rank matrix containing a set of smooth and linearly independent column vector fields spanning the null space of $\mathbf{C}_A$ i.e., $\mathbf{C}_A(\mathbf{q}_A)\mathbf{S}_A(\mathbf{q}_A) = \mathbf{0}$. Using (2), it is possible to find a velocity vector, $\boldsymbol{\nu}_A = [v, \dot{\theta}]^T$, such that

$$\dot{\mathbf{q}}_A = \mathbf{S}_A(\mathbf{q}_A)\boldsymbol{\nu}_A \tag{3}$$

where v and $\dot{\theta}$ are the forward and angular velocity inputs, while the detailed matrices involved in the model (1)-(3) are given by:

$$\mathbf{C}_A(\mathbf{q}_A) = \begin{bmatrix} S_\theta, \, -C_\theta, \, 0 \end{bmatrix} \quad ; \quad \mathbf{S}_A(\mathbf{q}_A) = \begin{bmatrix} C_\theta & 0 \\ S_\theta & 0 \\ 0 & 1 \end{bmatrix} \tag{4}$$

$$\mathbf{M}_A = \begin{bmatrix} m_0 & 0 & -am_0 S_\theta \\ 0 & m_0 & am_0 C_\theta \\ -am_0 S_\theta & am_0 C_\theta & a^2 m_0 + I_0 \end{bmatrix} \quad ; \quad \mathbf{V}_A = \begin{bmatrix} -am_0 \dot{\theta}^2 C_\theta \\ -am_0 \dot{\theta}^2 S_\theta \\ 0 \end{bmatrix} \tag{5}$$

$$\mathbf{E}_A = \begin{bmatrix} C_\theta / R & C_\theta / R \\ S_\theta / R & S_\theta / R \\ b/R & -b/R \end{bmatrix} \quad ; \quad \boldsymbol{\tau}_A = \begin{bmatrix} \tau_r \\ \tau_l \end{bmatrix} \tag{6}$$

where $S_\theta = \sin\theta$, $C_\theta = \cos\theta$, m_0 is the mass of the mobile base, I_0 is the moment of inertia of the mobile base, R is the radius of the wheels and b is defined as the half distance between the two wheels of the mobile base.

In order to incorporate the model of the manipulator arm, we consider a *two-link* planar arm moving in the horizontal plane. The manipulator arm is mounted on the mobile base at P_1 (see Figure 1). In this work, we adopt the sort of planar manipulator models used in the works of Ryu and Agrawal [21][22]. These manipulators are characterized as follows:

- m_i denotes the mass of the link i, l_i represents the length of the link and l_{ci} illustrates the distance of the center of mass of link i from joint i.
- The center of mass of the second link is at the second joint axis (i.e. $l_{c2} = 0$); the center of mass of the first and second links together is at the first joint (i.e. $m_1 l_{c1} + m_2 l_1 = 0$). Such inertia distribution can be achieved through counterbalancing (see [20] for details).
- In the manipulator arm considered in this work, the first joint is actuated with a torque input τ_1 while the second joint is unactuated but attached

to the first joint via a torsion spring. The spring constant will be specified as k_2.

Bearing in mind the previous assumptions, the dynamics of the manipulator arm can be derived using Lagrange's formulation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{Q}_B \tag{7}$$

with $L = K - V$ being the Lagrangian function, K and V being the kinetic and potential energies of the manipulator arm, and $\mathbf{Q}_B$ denotes the vector of generalized forces on the manipulator arm which is defined by the following expression:

$$\mathbf{Q}_B = \begin{bmatrix} \mathbf{R} \\ \boldsymbol{\tau}_B \end{bmatrix} \tag{8}$$

where $\boldsymbol{\tau}_B = [\tau_1, 0]^T$ and τ_1 denotes the torque input at joint 1. The potential energy of the manipulator arm, V, is assumed to be zero and the kinetic energy of the manipulator arm considered is represented using the following expression:

$$K = \sum_{i=1}^{i=2} \frac{1}{2}\left[m_i \mathbf{v}_{ci}^T \mathbf{v}_{ci} + I_i \boldsymbol{\omega}_i^T \boldsymbol{\omega}_i\right] \tag{9}$$

where m_i is the mass of link i, $\mathbf{v}_{ci}$ is the velocity at the center of mass of link i, I_i is the moment of inertia of link i and $\boldsymbol{\omega}_i$ is the inertial angular velocity of link i, expressed as

$$\omega_i = \left(\dot{\theta} + \sum_{k=1}^{i} \dot{\theta}_k\right)\mathbf{z} \tag{10}$$

where $\mathbf{z}$ is the unit vector along an axis normal to the horizontal plane of motion. The equations of motion of the manipulator can be written in the following block form:

$$\underbrace{\begin{bmatrix} \mathbf{M}_{B11} & \mathbf{M}_{B12} \\ \mathbf{M}_{B21} & \mathbf{M}_{B22} \end{bmatrix}}_{\mathbf{M}_B(\mathbf{q})} \underbrace{\begin{bmatrix} \ddot{\mathbf{q}}_A \\ \ddot{\mathbf{q}}_B \end{bmatrix}}_{\ddot{\mathbf{q}}} + \underbrace{\begin{bmatrix} \mathbf{V}_{B1} \\ \mathbf{V}_{B2} \end{bmatrix}}_{\mathbf{B}_B(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}} + \underbrace{\begin{bmatrix} \mathbf{G}_{B1} \\ \mathbf{G}_{B2} \end{bmatrix}}_{\mathbf{G}_B(\mathbf{q})} = \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{I}_2 \end{bmatrix} \boldsymbol{\tau}_B + \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix}}_{\mathbf{Q}_B} \tag{11}$$

where $\mathbf{M}_B(\mathbf{q})$ is a positive definite inertia matrix, $\mathbf{B}_B(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$ represents the vector of Coriolis and centripetal torques, $\mathbf{G}_B(\mathbf{q})$ denotes the vector of gravitational torques and $\mathbf{I}_2$ represents the identity matrix of dimension 2. Since the manipulator arm operates in a horizontal plane, $\mathbf{G}_B$ is assumed to be zero.

If the dynamic models of the mobile base and the manipulator arm are now merged, one obtains

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{G}_B(\mathbf{q}) = \mathbf{E}(\mathbf{q})\boldsymbol{\tau} - \begin{bmatrix} \mathbf{C}_A^T \boldsymbol{\lambda} \\ \mathbf{0} \end{bmatrix} \tag{12}$$

$$\dot{\mathbf{q}} = \mathbf{S}(\mathbf{q})\boldsymbol{\nu} \tag{13}$$

where

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} \mathbf{M}_A + \mathbf{M}_{B11} & \mathbf{M}_{B12} \\ \mathbf{M}_{B12}^T & \mathbf{M}_{B22} \end{bmatrix}, \qquad \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} \mathbf{V}_A + \mathbf{V}_{B1} \\ \mathbf{V}_{B2} \end{bmatrix} \tag{14}$$

$$\boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{\tau}_A \\ \boldsymbol{\tau}_B \end{bmatrix}, \qquad \mathbf{E}(\mathbf{q}) = \begin{bmatrix} \mathbf{E}_A & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 \end{bmatrix}, \qquad \mathbf{S}(\mathbf{q}) = \begin{bmatrix} \mathbf{S}_A & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 \end{bmatrix}, \qquad \boldsymbol{\nu} = \begin{bmatrix} \boldsymbol{\nu}_A \\ \dot{\mathbf{q}}_B \end{bmatrix} \tag{15}$$

In order to solve the dynamic problem, it is necessary to eliminate the vector of Lagrange multipliers. Differentiating both sizes in (13) gives $\ddot{\mathbf{q}} = \dot{\mathbf{S}}\boldsymbol{\nu} + \mathbf{S}\dot{\boldsymbol{\nu}}$, and substituting in (12) while multiplying the result by $\mathbf{S}^T$, leads to the following dynamic model:

$$\mathbf{A}(\mathrm{q})\dot{\boldsymbol{\nu}} + \mathbf{D}(\mathrm{q},\boldsymbol{\nu}) + \mathbf{G}(\mathrm{q}) = \mathbf{S}^T(\mathrm{q})\mathbf{E}(\mathrm{q})\boldsymbol{\tau} = \underbrace{\begin{bmatrix} \mathbf{S}_A^T\mathbf{E}_A & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 \end{bmatrix}}_{\mathbf{J}} \underbrace{\begin{bmatrix} \boldsymbol{\tau}_A \\ \boldsymbol{\tau}_B \end{bmatrix}}_{\boldsymbol{\tau}} \tag{16}$$

where $\mathbf{A}(\mathbf{q}) = \mathbf{S}^T\mathbf{M}\mathbf{S}$, $\mathbf{D}(\mathbf{q},\boldsymbol{\nu}) = \mathbf{S}^T\mathbf{M}\dot{\mathbf{S}}\boldsymbol{\nu} + \mathbf{S}^T\mathbf{V}$ and $\mathbf{G}(\mathbf{q}) = \mathbf{S}^T\mathbf{G}_B = [\mathbf{0}^T, \mathbf{G}_{B2}^T]^T$. From all previous considerations, the dynamic model of the robotic manipulator is the following:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{12} & a_{22} & a_3 & a_4 \\ 0 & a_3 & a_3 & a_4 \\ 0 & a_4 & a_4 & a_4 \end{bmatrix}}_{\mathbf{A}(\mathrm{q})} \underbrace{\begin{bmatrix} \dot{v} \\ \ddot{\theta} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}}_{\dot{\boldsymbol{\nu}}} + \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ 0 \\ 0 \end{bmatrix}}_{\mathrm{D}(\mathrm{q},\boldsymbol{\nu})} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ k_2\theta_2 \end{bmatrix}}_{\mathrm{G}(\mathrm{q})} = \underbrace{\begin{bmatrix} j_{11} & j_{12} & 0 & 0 \\ j_{21} & j_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{J}} \underbrace{\begin{bmatrix} \tau_r \\ \tau_l \\ \tau_1 \\ 0 \end{bmatrix}}_{\boldsymbol{\tau}} \tag{17}$$

where:

$$\begin{aligned} & a_{11} = m_0 + m_1 + m_2; \quad a_{12} = 0; \quad a_{22} = \alpha_0 + \alpha_1 + I_0 + I_1 + I_2 \\ & a_3 = \alpha_1 + I_1 + I_2; \quad a_4 = I_2; \quad d_1 = -\zeta_0\dot{\theta}^2; \quad d_2 = \zeta_0\dot{\theta}v \\ & j_{11} = 1/R; \quad j_{12} = 1/R; \quad j_{21} = b/R; \quad j_{22} = -b/R \\ & \zeta_0 = m_0 a + (m_1 + m_2)d; \quad \alpha_0 = m_0 a^2 + (m_1 + m_2)d^2; \quad \alpha_1 = m_1 l_{c1}^2 + m_2 l_1^2 \end{aligned} \tag{18}$$

where m_1 and I_1 are, respectively, the mass and the moment of inertia of the first link of the arm and m_2 and I_2 are the mass and the moment of inertia of the second link of the arm (adapted from [21][22]).

2.2 Flatness of the system

According to the theory of differential flatness [17], a dynamic system, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u})$, with $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^m$, is said to be differentially flat if there exist, m, differentially independent variables called *flat outputs* (differentially independent meaning that they are not related by differential equations), which are functions of the state vector and, possibly, of a finite number of time derivatives of the state vector (i.e., derivatives of the inputs may be involved in their definition), such that *all* system variables (states, inputs, outputs, and functions of these variables) can, in turn, be expressed as functions of the flat outputs and of a finite number of their time derivatives. This parameterization establishes a one-to-one mapping from the states and the inputs to the flat outputs. Since the number of flat outputs is equal to the number of control inputs [17], this establishes the full-state controllability of the system. Contrary to unwarranted belief, flatness is not simply another way in which to carry out feedback linearization. It is, in fact, a structural property of the system that allows all the salient features which are needed for the application of a particular feedback controller design technique (such as backstepping, passivity, sliding and, of course, feedback linearization) to be established. It is a property that readily trivializes the exact linearization problem in a nonlinear system, whether or not the system is multivariable, and whether or not it is affine in the control inputs. Moreover, flatness directly applies to any nonlinear system, regardless of the nonlinear, or affine, nature of the control inputs in the system equations. A constraint-satisfying desired trajectory can now be planned using a variety of time functions matching the constraint conditions in the flat output space. The flat outputs, being devoid of any zero dynamics, completely guarantee the total internal stability of the system states. All these aspects facilitate a unified treatment for both, stabilization and trajectory tracking tasks, within a common framework.

Taking into account the works developed by Agrawal and his co-workers [22][23], if the inertia distribution within the manipulator arm is properly chosen, mobile manipulator systems can be made to be differentially flat. The proposed system is differentially flat with flat outputs given by the components of the three vector:

$$\mathbf{L} = [x, y, F]^T \tag{19}$$

with (x, y) being the horizontal and vertical position of the point O in a coordinate frame and F representing the sum of the orientation of the mobile base θ and all the relative joint angles θ_i of the manipulator arm, i.e., $F = \theta + \sum_{i=1}^{2} \theta_i$. We shall now demonstrate the parameterizations of all the system variables in terms of the three vector $\mathbf{L}$ and of a finite number of derivatives of its components.

Proposition 1: The wheeled mobile underactuated manipulator model given in (17) is differentially flat, with flat outputs given by the three vector $\mathbf{L} = [x, y, F]^T$, i.e., all system variables in (17) can be differentially parameterized solely in terms of x, y, F, and a finite number of their time derivatives. Their expressions are:

$$\theta = \arctan\left(\frac{\dot{y}}{\dot{x}}\right) \tag{20}$$

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} \tag{21}$$

$$\theta_2 = -\frac{a_4}{k_2}\ddot{F} \tag{22}$$

$$\theta_1 = F - \arctan\left(\frac{\dot{y}}{\dot{x}}\right) - \frac{a_4}{k_2}\ddot{F} \tag{23}$$

$$\tau_a = \frac{\ddot{x}\dot{x} + \ddot{y}\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \tag{24}$$

$$\tau_b = \frac{\left(-x^{(3)}\dot{y} + y^{(3)}\dot{x}\right)\left(\dot{x}^2 + \dot{y}^2\right) - 2\left(\ddot{x}\dot{x} + \ddot{y}\dot{y}\right)\left(\dot{x}\ddot{y} - \dot{y}\ddot{x}\right)}{\left(\dot{x}^2 + \dot{y}^2\right)^2} \tag{25}$$

$$\tau_c = \frac{\left(x^{(3)}\dot{y} - y^{(3)}\dot{x}\right)\left(\dot{x}^2 + \dot{y}^2\right) + 2\left(\ddot{x}\dot{x} + \ddot{y}\dot{y}\right)\left(\dot{x}\ddot{y} - \dot{y}\ddot{x}\right)}{\left(\dot{x}^2 + \dot{y}^2\right)^2} + \frac{a_4}{k_2}F^{(4)} + \ddot{F} \tag{26}$$

$$\begin{bmatrix} \tau_r \\ \tau_l \\ \tau_1 \end{bmatrix} = \begin{bmatrix} j_{11} & j_{12} & 0 \\ j_{21} & j_{22} & -1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \left[\begin{bmatrix} a_{11} & a_{12} & 0 \\ 0 & (a_{22} - a_4) & (a_3 - a_4) \\ 0 & (a_3 - a_4) & (a_3 - a_4) \end{bmatrix} \begin{bmatrix} \tau_a \\ \tau_b \\ \tau_c \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 - k_2\theta_2 \\ -k_2\theta_2 \end{bmatrix}\right] \tag{27}$$

Proof: From the last row of expression (17) it is obtained:

$$\ddot{\theta}_2 = -\frac{k_2}{a_4}\theta_2 - (\ddot{\theta} + \ddot{\theta}_1) \tag{28}$$

If the previous expression in the first three rows of (17) is substituted and the terms are rearranged, the following is obtained:

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ 0 & (a_{22} - a_4) & (a_3 - a_4) \\ 0 & (a_3 - a_4) & (a_3 - a_4) \end{bmatrix} \begin{bmatrix} \dot{v} \\ \ddot{\theta} \\ \ddot{\theta}_1 \end{bmatrix} = \begin{bmatrix} -d_1 \\ -d_2 + k_2\theta_2 \\ k_2\theta_2 \end{bmatrix} + \begin{bmatrix} j_{11} & j_{12} & 0 \\ j_{21} & j_{22} & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tau_r \\ \tau_l \\ \tau_1 \end{bmatrix} \tag{29}$$

The following virtual input vector can be defined from (29):

$$\begin{bmatrix} \tau_a \\ \tau_b \\ \tau_c \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ 0 & (a_{22}-a_4) & (a_3-a_4) \\ 0 & (a_3-a_4) & (a_3-a_4) \end{bmatrix}^{-1} \left[\begin{bmatrix} -d_1 \\ -d_2+k_2\theta_2 \\ k_2\theta_2 \end{bmatrix} + \begin{bmatrix} j_{11} & j_{12} & 0 \\ j_{21} & j_{22} & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tau_r \\ \tau_l \\ \tau_1 \end{bmatrix} \right] \tag{30}$$

This yields the following simplified dynamics:

$$\begin{bmatrix} \dot{v} \\ \ddot{\theta} \\ \ddot{\theta}_1 \end{bmatrix} = \begin{bmatrix} \tau_a \\ \tau_b \\ \tau_c \end{bmatrix} \tag{31}$$

From expression (13) we now readily obtain

$$\dot{x} = vC_\theta; \quad \dot{y} = vS_\theta \tag{32}$$

and when operating with (32) the following relations are yielded

$$v = \sqrt{\dot{x}^2 + \dot{y}^2}; \quad \theta = \arctan\left(\frac{\dot{y}}{\dot{x}}\right) \tag{33}$$

If the equations given in (33) are now differentiated with regard to the time, and taking into account (32), the following result is yielded:

$$\begin{aligned} \dot{v} &= \frac{\dot{x}\ddot{x} + \dot{y}\ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \ddot{x}C_\theta + \ddot{y}S_\theta \\ \dot{\theta} &= \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2} = \frac{\ddot{y}C_\theta - \ddot{x}S_\theta}{v} \end{aligned} \tag{34}$$

Similarly, upon operating with (34) we achieve

$$\begin{aligned} \ddot{x} &= \dot{v}C_\theta - v\dot{\theta}S_\theta; \\ \ddot{y} &= \dot{v}S_\theta + v\dot{\theta}C_\theta \end{aligned} \tag{35}$$

If the expressions (34) are differentiated with regard to the time and the terms are rearranged:

$$\ddot{v} = x^{(3)}C_\theta + y^{(3)}S_\theta + \dot{\theta}^2 v \qquad (36)$$
$$\ddot{\theta} = \frac{-x^{(3)}S_\theta + y^{(3)}C_\theta - 2\dot{\theta}\dot{v}}{v}$$

On the other hand, expression (28) can be written has:

$$\theta_2 = -\frac{a_4}{k_2}\underbrace{\left(\ddot{\theta} + \ddot{\theta}_1 + \ddot{\theta}_2\right)}_{\ddot{F}} = -\frac{a_4}{k_2}\ddot{F} \qquad (37)$$

Taking into account that $F = \theta + \theta_1 + \theta_2$, we now obtain

$$\theta_1 = F - \theta - \theta_2 = F - \arctan\left(\frac{\dot{y}}{\dot{x}}\right) + \frac{a_4}{k_2}\ddot{F} \qquad (38)$$

If the expression (38) is differentiated twice with regard to the time and the terms are rearranged:

$$\ddot{\theta}_1 = \ddot{F} - \ddot{\theta} + \frac{a_4}{k_2}F^{(4)} = \frac{x^{(3)}S_\theta - y^{(3)}C_\theta + 2\dot{\theta}\dot{v}}{v} + \frac{a_4}{k_2}F^{(4)} + \ddot{F} \qquad (39)$$

Then, considering (31), (35), (36) and (39) the following result is achieved:

$$\tau_a = \ddot{x}C_\theta + \ddot{y}S_\theta = \frac{\ddot{x}\dot{x} + \ddot{y}\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \qquad (40)$$
$$\tau_b = \frac{-x^{(3)}S_\theta + y^{(3)}C_\theta - 2\dot{v}\dot{\theta}}{v} = \frac{\left(-x^{(3)}\dot{y} + y^{(3)}\dot{x}\right)\left(\dot{x}^2 + \dot{y}^2\right) - 2\left(\ddot{x}\dot{x} + \ddot{y}\dot{y}\right)\left(\dot{x}\ddot{y} - \dot{y}\ddot{x}\right)}{\left(\dot{x}^2 + \dot{y}^2\right)^2} \qquad (41)$$
$$\tau_c = \frac{x^{(3)}S_\theta - y^{(3)}C_\theta + 2\dot{v}\dot{\theta}}{v} + \frac{a_4}{k_2}F^{(4)} + \ddot{F} = \frac{\left(x^{(3)}\dot{y} - y^{(3)}\dot{x}\right)\left(\dot{x}^2 + \dot{y}^2\right) + 2\left(\ddot{x}\dot{x} + \ddot{y}\dot{y}\right)\left(\dot{x}\ddot{y} - \dot{y}\ddot{x}\right)}{\left(\dot{x}^2 + \dot{y}^2\right)^2} + \frac{a_4}{k_2}F^{(4)} + \ddot{F} \qquad (42)$$

Finally, using (30) one obtains the following result

$$\begin{bmatrix} \tau_r \\ \tau_l \\ \tau_1 \end{bmatrix} = \begin{bmatrix} j_{11} & j_{12} & 0 \\ j_{21} & j_{22} & -1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \left[\begin{bmatrix} a_{11} & a_{12} & 0 \\ 0 & (a_{22} - a_4) & (a_3 - a_4) \\ 0 & (a_3 - a_4) & (a_3 - a_4) \end{bmatrix}\begin{bmatrix} \tau_a \\ \tau_b \\ \tau_c \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 - k_2\theta_2 \\ -k_2\theta_2 \end{bmatrix}\right] \qquad (43)$$

$\square$

From expressions (40), (41) and (42) it is observed the lack of invertibility of the relation between the virtual control input vector, $[\tau_a, \tau_b, \tau_c]^T$, and the flat output highest derivatives. This reveals an obstacle in the virtual input τ_a to achieve static feedback linearization and points to the need for a first order dynamic extension of the virtual control input τ_a in order to exactly linearize the system (see [17] for details on the use of dynamic feedback). This yields:

$$\dot{\tau}_a = x^{(3)} C_\theta + y^{(3)} S_\theta + \dot{v}\dot{\theta}^2 \tag{44}$$

Finally, upon merging expressions (41), (42) and (44) we achieve the following input-to-highest derivative of flat outputs relations:

$$\begin{bmatrix} \dot{\tau}_a \\ \tau_b \\ \tau_c \end{bmatrix} = \underbrace{\begin{bmatrix} C_\theta & S_\theta & 0 \\ -\frac{S_\theta}{v} & \frac{C_\theta}{v} & 0 \\ \frac{S_\theta}{v} & -\frac{C_\theta}{v} & \frac{a_4}{k_2} \end{bmatrix}}_{\mathcal{N}^{-1}} \begin{bmatrix} x^{(3)} \\ y^{(3)} \\ F^{(4)} \end{bmatrix} + \begin{bmatrix} \dot{v}\dot{\theta}^2 \\ -\frac{2\dot{v}\dot{\theta}}{v} \\ \frac{2\dot{v}\dot{\theta}}{v} + \ddot{F} \end{bmatrix} \tag{45}$$

2.3 Simplified Model

The key step in our developments is based on the fact that the flat output dynamics of the robotic platform (45) may be significantly simplified to the following perturbed, non-phenomenological, simplified model:

$$\begin{bmatrix} x^{(3)} \\ y^{(3)} \\ F^{(4)} \end{bmatrix} = \underbrace{\begin{bmatrix} C_\theta & -vS_\theta & 0 \\ S_\theta & vC_\theta & 0 \\ 0 & \frac{k_2}{a_4} & \frac{k_2}{a_4} \end{bmatrix}}_{\mathcal{N}} \begin{bmatrix} \dot{\tau}_a \\ \tau_b \\ \tau_c \end{bmatrix} + \underbrace{\begin{bmatrix} \varphi_x \\ \varphi_y \\ \varphi_F \end{bmatrix}}_{\boldsymbol{\varphi}(t)} \tag{46}$$

where $\boldsymbol{\varphi}(t) = [\varphi_x, \varphi_y, \varphi_F]^T$ involves all external disturbances and the effect of the nonlinearities affecting the system behavior, which is here regarded as an unknown but uniformly absolutely bounded disturbance input that needs to be estimated on-line by means of an observer and, subsequently, canceled from the simplified system dynamics via feedback in order to regulate the flat output vector, $\mathbf{L} = [x, y, F]^T$ towards the desired reference trajectories $\mathbf{L}^* = [x^*, y^*, F^*]^T$. It is assumed that the terms a_2 and k_2 given in expression (46) are known. A key property of the flat output vector $\mathbf{L}$ that allows us to comfortably consider this important system simplification is represented by the fact that the flat output vector is devoid of any zero dynamics (see [29] for a study of this important concept, in the general setting of nonlinear systems).

2.4 Problem Formulation

The following problem formulation is stated for the mobile manipulator studied in this work:

Given a flat output vector of reference trajectories, $\mathbf{L}^*(t) = [x^*(t), y^*(t), F^*(t)]^T$*, devise a linear multi-input output feedback controller that suitably cancels, even if in an approximate manner, the vector of coupling nonlinearities,* $\boldsymbol{\varphi}(t) = [\varphi_x, \varphi_y, \varphi_F]^T$*, and forces the flat output tracking error vector dynamics,* $\mathbf{e}_L = [e_x, e_y, e_F]^T = [x - x^*, y - y^*, F - F^*]^T$*, to exhibit a closed loop, predominantly linear, asymptotically stable convergent behavior so that* *the tracking error trajectories are ultimately confined to a small as desired neighborhood of the origin of the tracking error phase space.*

3 LINEAR GPI OBSERVER-BASED CONTROL OF NONLINEAR SYSTEMS

This section presents some generalities as regards the GPI observer-based output feedback control approach to the solution of the trajectory tracking problem for a perturbed linear system and its application to the trajectory tracking control of underactuated mobile manipulators.

3.1 Mathematical Framework

Consider the following perturbed nonlinear single-input single input-output, smooth, nonlinear system,

$$y^{(n)} = \Psi(t, y, \dot{y}, \dots, y^{(n-1)}) + \Phi(t, y)u + \zeta(t) \tag{47}$$

The unperturbed system, $(\zeta(t) \equiv 0)$ is evidently flat, as all the variables are expressible as differential functions of the flat output y. We assume that the exogenous perturbation $\zeta(t)$ is uniformly absolutely bounded, i.e., that it is a L_∞ scalar function. We similarly assume that for all bounded solutions, $y(t)$, of (47), obtained by means of a suitable control input u, the additive, endogenous, perturbation input, $\Psi(t, y(t), \dot{y}(t), \dots, y^{(n-1)})$, viewed as a time signal, is uniformly absolutely bounded. We also assume that the nonlinear gain function $\Phi(t, y(t))$ is L_∞ and is uniformly bounded away from zero, i.e., there exists a strictly positive constant μ such that

$$\inf_t |\Phi(t, y(t))| \geq \mu \tag{48}$$

for all smooth, bounded solutions, $y(t)$, of (47) obtained with a suitable control input u. Although the results below can be extended when the input gain function Φ depends on the time derivatives of y, we let Φ, motivated by the underactuated wheeled mobile manipulator study to be presented below, be an explicit function of time and of the measured flat output y. This is equivalent to saying that $\Phi(t, y(t))$ is perfectly known.

We consider the following problem: *Given a desired flat output reference trajectory, $y^*(t)$, devise a linear output feedback controller for system (47) so that regardless of the endogenous perturbation signal $\Psi(t, y(t), \dot{y}(t), \ldots, y^{(n-1)})$ and of the exogenous perturbation input $\zeta(t)$, the flat output y tracks the desired reference signal $y^*(t)$ even if in an approximate fashion. This approximate character specifically means that the tracking error, $e(t) = y - y^*(t)$, and its first, n, time derivatives, globally asymptotically exponentially converge towards a small as desired neighborhood of the origin in the reference trajectory tracking error phase space.*

The solution to the problem is achieved in an entirely linear fashion if the nonlinear model (47) is conceptually considered as the following *linear perturbed system*

$$y^{(n)} = \nu + \xi(t) \tag{49}$$

where $\nu = \Phi(t, y)u$, and $\xi(t) = \Psi(t, y(t), \dot{y}(t), \ldots, y^{(n-1)}) + \zeta(t)$.

Consider the following preliminary result:

Proposition 2: The unknown perturbation vector of time signals, $\xi(t)$, in the simplified tracking error dynamics (49) is *observable* in the sense of Diop and Fliess (see [30] for details).

Proof The proof of this fact is immediate after writing (49) as:

$$\xi(t) = y^{(n)} - \nu = y^{(n)} - \Phi(t, y)u \tag{50}$$

i.e., $\xi(t)$ can be written in terms of the output vector y, a finite number of its time derivatives and the control input u. Hence, $\xi(t)$ is observable. □

Remark 1:: This means, in particular, that if $\xi(t)$ is bestowed with an exact linear model; an exact asymptotic estimation of $\xi(t)$ can be asymptotically estimated with the help of a linear observer subject to nonlinear input injection through the known gain. If, on the other hand, the linear model is only approximately locally valid, then the estimation obtained via a linear observer is asymptotically convergent towards an equally approximately locally valid estimate.

We assume that the perturbation input $\xi(t)$ may be locally modeled as a $p-1$-th degree time polynomial z_1 plus a residual term, $r(t)$, i.e.,

$$\xi(t) = z_1 + r(t) = a_0 + a_1 t + \ldots + a_{p-1} t^{p-1} + r(t) \quad \forall t \tag{51}$$

The time polynomial model, $\frac{d^p z_1(t)}{dt^p} = 0$, (also called: a Taylor polynomial) is invariant with regard to time shifts and it defines a family of $p-1$ degree Taylor polynomials with arbitrary real coefficients. We incorporate z_1 as an internal model of the additive perturbation input (see [31]). The perturbation model, z_1, will acquire a *self updating* character when incorporated as part of a linear asymptotic observer whose estimation error is forced to converge to a small vicinity of zero. As a consequence of this, we may safely assume that the self-updating residual function, $r(t)$, and its time derivatives, say $r^{(p)}(t)$, are uniformly absolutely bounded. In order to state this precisely, let us use y_j to denote an estimate of $y^{(j-1)}$ for $j = 1, \ldots, n$. The following general result is obtained:

Theorem 1: The GPI observer-based dynamical feedback controller:

$$
\begin{aligned}
u = \frac{1}{\Phi(t,y)} & \left[[y^*(t)]^{(n)} - \sum_{j=0}^{n-1} \left(\kappa_j [y_j - (y^*(t))^{(j)}] \right) - \hat{\xi}(t) \right] \\
\hat{\xi}(t) &= z_1
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
\dot{y}_1 &= y_2 + \lambda_{p+n-1}(y - y_1) \\
\dot{y}_2 &= y_3 + \lambda_{p+n-2}(y - y_1) \\
&\vdots \\
\dot{y}_n &= \nu + z_1 + \lambda_p(y - y_1) \\
\dot{z}_1 &= z_2 + \lambda_{p-1}(y - y_1) \\
&\vdots \\
\dot{z}_{p-1} &= z_p + \lambda_1(y - y_1) \\
\dot{z}_p &= \lambda_0(y - y_1)
\end{aligned} \tag{53}
$$

asymptotically exponentially drives the tracking error phase variables $e_y^{(k)} = y^{(k)} - [y^*]^{(k)}$, $k = 0, 1, \ldots, n-1$ to an arbitrary small neighborhood of the origin, of the tracking error phase space, which can be made as small as desired from the appropriate choice of the controller gain parameters $\{\kappa_0, \ldots, \kappa_{n-1}\}$. Moreover, the estimation errors: $\tilde{e}^{(i)} = y^{(i)} - y_i$, $i = 0, \ldots, n-$

1 and the perturbation estimation error: $z_m - \xi^{(m-1)}(t)$, $m = 1, \ldots, p$ asymptotically exponentially converge towards a small as desired neighborhood of the origin of the reconstruction error space, which can be made as small as desired with the appropriate choice of the controller gain parameters $\{\lambda_0, \ldots, \lambda_{p+n-1}\}$.

Proof The proof is based on the fact that the estimation error $\tilde{e}$ satisfies the perturbed linear differential equation

$$\tilde{e}^{(p+n)} + \lambda_{p+n-1}\tilde{e}^{(p+n-1)} + \ldots + \lambda_0 \tilde{e} = r^{(p)}(t) \tag{54}$$

Since $r^{(p)}(t)$ is assumed to be uniformly absolutely bounded, then there exists coefficients λ_k such that $\tilde{e}$ converges to a small vicinity of zero, provided if the roots of the associated characteristic polynomial in the complex variable s,

$$s^{(p+n)} + \lambda_{p+n-1}s^{(p+n-1)} + \ldots + \lambda_1 s + \lambda_0 \tag{55}$$

are all located deep in the left half of the complex plane. The further away from the imaginary axis of the complex plane these roots are located, the smaller the neighborhood of the origin is in the estimation error phase space, in which the estimation error $\tilde{e}$ will remain ultimately bounded (see [32]). Clearly, if $\tilde{e}$ and its time derivatives converge to a neighborhood of the origin, then $z_j - \xi^j$, $j = 1, 2, \ldots$, also converge towards a small vicinity of zero.

The tracking error $e_y = y - y^*(t)$ evolves according to the following linear perturbed dynamics

$$e_y^{(n)} + \kappa_{n-1}e_y^{(n-1)} + \ldots + \kappa_0 e_y = \xi - \hat{\xi}(t) \tag{56}$$

By choosing the controller coefficients $\{\kappa_0, \ldots, \kappa_{n-1}\}$, so that the associated characteristic polynomial

$$s^{(n)} + \kappa_{n-1}s^{(n-1)} + \ldots + \kappa_1 s + \kappa_0 \tag{57}$$

exhibits its roots sufficiently far from the imaginary axis in the left half portion of the complex plane, the tracking error and its various time derivatives, are guaranteed to converge asymptotically exponentially towards a vicinity of the tracking error phase space. Note that, according to the observer's expected performance, the right-hand side of (56) is represented by a uniformly absolutely bounded signal that is already evolving on a small vicinity of the origin. The roots of (57) may therefore be located closer to the imaginary axis than those of (55). A more detailed proof of this theorem can be found in [33]. $\square$

Remark 2: The proposed GPI observer (53) is a high gain observer which is prone to exhibiting the *peaking* phenomena at the initial time. We use a suitable *clutch* to smooth out these transient peaking responses in all the observer variables that need to be used by the controller. This is accomplished using a factor function which smoothly interpolates between an initial value of zero and a final value of unity. We denote this clutching function as $s_f(t) \in [0, 1]$ and define it in the following (non-unique) way

$$s_f(t) = \begin{cases} 1 \text{ for } t > \varepsilon \\ \sin^q\left(\frac{\pi t}{2\varepsilon}\right) \text{ for } t \leq \varepsilon \end{cases} \tag{58}$$

where q is a suitably large positive even integer. Thus, for example, a *smoothing* variable function of $H_j(t)$ is yielded as $H_{js}(t) = H_j(t)s_f(t)$.

3.2 GPI observer based control of the Underactuated Wheeled Mobile Manipulators

Based on the reduced model (46) for two-wheeled mobile manipulators which has been substantially simplified by resorting to the original system representation and acknowledging the flatness based structural findings, we shall now deal with the design of the GPI observer based controller of the underactuated mobile manipulator presented. In this case, the outputs x and y are each relative degree three with a first order dynamic extension of the virtual input τ_a and the output F is relative degree 4. A GPI observer including a reasonable, self-updating, time-polynomial model[1] is considered for each unknown component, state dependent, disturbance input vector $\boldsymbol{\varphi}(t)$. For this internal model, we use for each component of $\boldsymbol{\varphi}(t)$ an unspecified element of a fifth order family of time-polynomials, denoted by $\boldsymbol{\varphi}_1^{(6)}(t) = [\varphi_{1x}^{(6)}, \varphi_{1y}^{(6)}, \varphi_{1F}^{(6)}]^T = \mathbf{0}$. The GPI observer based flat output feedback controller is then synthesized as follows:

$$\begin{bmatrix} \dot{\tau}_a \\ \tau_b \\ \tau_c \end{bmatrix} = \underbrace{\begin{bmatrix} C_{\hat{\theta}_s} & S_{\hat{\theta}_s} & 0 \\ -\frac{S_{\hat{\theta}_s}}{\hat{v}_s} & \frac{C_{\hat{\theta}_s}}{\hat{v}_s} & 0 \\ \frac{S_{\hat{\theta}_s}}{\hat{v}_s} & -\frac{C_{\hat{\theta}_s}}{\hat{v}_s} & \frac{a_4}{k_2} \end{bmatrix}}_{\mathcal{N}^{-1}(\dot{x},\dot{y})} \begin{bmatrix} \nu_x \\ \nu_y \\ \nu_F \end{bmatrix} \tag{59}$$

with

$$\begin{aligned} \nu_x &= -\hat{\varphi}_{1xs} + [x^*(t)]^{(3)} - \sum_{i=0}^{2} k_i^x \left(\hat{x}_s^{(i)} - [x^*]^{(i)}\right) \\ \nu_y &= -\hat{\varphi}_{1ys} + [y^*(t)]^{(3)} - \sum_{i=0}^{2} k_i^y \left(\hat{y}_s^{(i)} - [y^*]^{(i)}\right) \\ \nu_F &= -\hat{\varphi}_{1Fs} + [F^*(t)]^{(4)} - \sum_{i=0}^{3} k_i^F \left(\hat{F}_s^{(i)} - [F^*]^{(i)}\right) \end{aligned} \tag{60}$$

[1] Also known as Taylor Polynomial Model

where the quantities with subindex s are smoothing observer variables which are carried out by means of the following *clutching function*, avoiding possible large peaks in their high gain induced responses:

$$s_f(t) = \begin{cases} 1 \text{ for } t > \varepsilon \\ \sin^8\left(\frac{\pi t}{2\varepsilon}\right) \text{ for } t \leq \varepsilon \end{cases} \tag{61}$$

with $\epsilon = 3\,[s]$. Furthermore, the variables $\hat{x}^{(j)} = x_j$, $\hat{y}^{(j)} = y_j$, $j = 0, 1, 2$ and $\hat{F}^{(k)} = F_k$, $k = 0, 1, \ldots, 4$ are generated by:

$$\begin{aligned}
\dot{x}_0 &= x_1 + \lambda_7^x(x - x_0) \\
\dot{x}_1 &= x_2 + \lambda_6^x(x - x_0) \\
\dot{x}_2 &= \dot{\tau}_a C_{\hat{\theta}_s} - \tau_b v_s S_{\hat{\theta}_s} + \varphi_{1x} + \lambda_5^x(x - x_0) \\
\dot{\varphi}_{1x} &= \varphi_{2x} + \lambda_4^x(x - x_0) \\
\dot{\varphi}_{2x} &= \varphi_{3x} + \lambda_3^x(x - x_0) \\
\dot{\varphi}_{3x} &= \varphi_{4x} + \lambda_2^x(x - x_0) \\
\dot{\varphi}_{4x} &= \varphi_{5x} + \lambda_1^x(x - x_0) \\
\dot{\varphi}_{5x} &= \lambda_0^x(x - x_0)
\end{aligned} \tag{62}$$

$$
\begin{aligned}
\dot{y}_0 &= y_1 + \lambda_7^y (y - y_0) \\
\dot{y}_1 &= y_2 + \lambda_6^y (y - y_0) \\
\dot{y}_2 &= \dot{\tau}_a S_{\hat{\theta}_s} + \tau_b v_s C_{\hat{\theta}_s} + \varphi_{1y} + \lambda_5^y (y - y_0) \\
\dot{\varphi}_{1y} &= \varphi_{2y} + \lambda_4^y (y - y_0) \\
\dot{\varphi}_{2y} &= \varphi_{3y} + \lambda_3^y (y - y_0) \\
\dot{\varphi}_{3y} &= \varphi_{4y} + \lambda_2^y (y - y_0) \\
\dot{\varphi}_{4y} &= \varphi_{5y} + \lambda_1^y (y - y_0) \\
\dot{\varphi}_{5y} &= \lambda_0^y (y - y_0)
\end{aligned}
\tag{63}
$$

$$
\begin{aligned}
\dot{F}_0 &= F_1 + \lambda_8^F (F - F_0) \\
\dot{F}_1 &= F_2 + \lambda_7^F (F - F_0) \\
\dot{F}_2 &= F_3 + \lambda_6^F (F - F_0) \\
\dot{F}_3 &= \frac{k_2}{a_4} (\tau_b + \tau_c) + \varphi_{1F} + \lambda_5^F (F - F_0) \\
\dot{\varphi}_{1F} &= \varphi_{2F} + \lambda_4^F (F - F_0) \\
\dot{\varphi}_{2F} &= \varphi_{3F} + \lambda_3^F (F - F_0) \\
\dot{\varphi}_{3F} &= \varphi_{4F} + \lambda_2^F (F - F_0) \\
\dot{\varphi}_{4F} &= \varphi_{5F} + \lambda_1^F (F - F_0) \\
\dot{\varphi}_{5F} &= \lambda_0^F (F - F_0)
\end{aligned}
\tag{64}
$$

The estimation error dynamics $\tilde{e}_x = x - x_0$, $\tilde{e}_y = y - y_0$ and $\tilde{e}_F = F - F_0$ evolve in accordance with the linear perturbed dynamics:

$$\begin{aligned}
&\tilde{e}_x^{(8)} + \lambda_7^x \tilde{e}_x^{(7)} + \ldots + \lambda_1^x \dot{\tilde{e}}_x + \lambda_0^x \tilde{e}_x = \varphi_x^{(6)}(t) \\
&\tilde{e}_y^{(8)} + \lambda_7^y \tilde{e}_y^{(7)} + \ldots + \lambda_1^y \dot{\tilde{e}}_y + \lambda_0^y \tilde{e}_y = \varphi_y^{(6)}(t)(t) \\
&\tilde{e}_F^{(9)} + \lambda_8^F \tilde{e}_F^{(8)} + \ldots + \lambda_1^F \dot{\tilde{e}}_F + \lambda_0^F \tilde{e}_F = \varphi_F^{(6)}(t)
\end{aligned} \tag{65}$$

Clearly, if $\varphi_x^{(6)}(t)$, $\varphi_y^{(6)}(t)$ and $\varphi_F^{(6)}(t)$ are uniformly absolutely bounded, and if the choice of the coefficients $\{\lambda_7^x, \ldots, \lambda_0^x\}$, $\{\lambda_7^y, \ldots, \lambda_0^y\}$ and $\{\lambda_8^F, \ldots, \lambda_0^F\}$ is made in such a way that the roots of the dominant characteristic polynomials,

$$\begin{aligned}
&\hat{p}_x(s) = s^{(8)} + \lambda_7^x s^{(7)} + \ldots + \lambda_1^x s + \lambda_0^x \\
&\hat{p}_y(s) = s^{(8)} + \lambda_7^y s^{(7)} + \ldots + \lambda_1^y s + \lambda_0^y \\
&\hat{p}_F(s) = s^{(9)} + \lambda_8^F s^{(8)} + \ldots + \lambda_1^F s + \lambda_0^F
\end{aligned} \tag{66}$$

are defined as a 8-th (for $\hat{p}_x(s)$ and $\hat{p}_y(s)$) and 9-th (for $\hat{p}_F(s)$) degree Hurwitz polynomial and their roots are located sufficiently far from the imaginary axis, in the left half of the complex plane, then the trajectories of the estimation errors, $\tilde{e}_x$, $\tilde{e}_y$ and $\tilde{e}_F$, and of their time derivatives, will converge to a small neighborhood of the origin of the phase space of the observer estimation error. The further away the roots are located in the left half of the complex plane, the smaller the radius of the disk representing the neighborhood around the origin of the estimation error phase space will be.

The closed loop tracking errors $e_x = x - x^*$, $e_y = y - y^*$ and $e_F = F - F^*$ satisfy the following predominantly linear dynamics:

$$
\begin{aligned}
& e_x^{(3)} + k_2^x \ddot{e}_x + k_1^x \dot{e}_x + k_0^x e_x = \varphi_x(t) - \hat{\varphi}_x(t) + \varrho_x(t) \\
& e_y^{(3)} + k_2^y \ddot{e}_y + k_1^y \dot{e}_y + k_0^y e_y = \varphi_y(t) - \hat{\varphi}_y(t) + \varrho_y(t) \\
& e_F^{(4)} + k_3^F e_F^{(3)} + k_2^F e_F^{(2)} + k_1^F \dot{e}_F + k_0^F e_F = \varphi_F(t) - \hat{\varphi}_F(t) + \varrho_z(t)
\end{aligned}
\tag{67}
$$

where $\boldsymbol{\varrho}(t) = [\varrho_x(t), \varrho_y(t), \varrho_z(t)]^T$ in expression (67), depicts the effect of the small flat output phase variable estimation errors, generated by the observer, and the effects of the on line disturbance signal estimation error and the differences $\varphi_x(t) - \hat{\varphi}_x$, $\varphi_y(t) - \hat{\varphi}_y$ and $\varphi_F(t) - \hat{\varphi}_F$ produce reference trajectory tracking errors $e_x = x - x^*$, $e_y = y - y^*$ and $e_F = F - F^*$, which also asymptotically exponentially converge towards a small vicinity of the origin of the tracking error space. The design coefficients k_i^x, k_i^y and k_i^F are chosen so as to render the closed loop characteristic polynomials,

$$
\begin{aligned}
p_x(s) &= s^3 + k_2^x s^2 + k_1^x s + k_0^x \\
p_y(s) &= s^3 + k_2^y s^2 + k_1^y s + k_0^y \\
p_F(s) &= s^4 + k_3^F s^3 + k_2^F s^2 + k_1^F s + k_0^F
\end{aligned}
\tag{68}
$$

into Hurwitz polynomials with desirable root locations. It is intuitively clear that the closed loop dynamics in expression (68) is less severely affected by the uncertainties than the corresponding dynamics of the observer estimation error. This fact results in smaller magnitudes of the feedback gains k_i^x, k_i^y and k_i^F than those used for the design of the GPI observers. Finally, we have the following result:

Proposition 3: Given a smooth vector of desired reference trajectories for the components of the flat output vector, $\mathbf{L}^*(t) = [x^*(t), y^*(t), F^*(t)]^T$, and provided the observers' and the controllers' constant gains that appear in (66), (68) are chosen so that the roots of the corresponding closed loop characteristic polynomials are chosen deep in the left half of the complex plane, then the GPI observer based linear feedback controllers given by equations: (59)-(64), produce a set of perturbed closed loop flat outputs tracking error dynamics whose trajectories converge, in an asymptotically exponentially dominated manner, to a small as desired neighborhood of the origins of the flat output tracking error phase spaces. Moreover, the flat output phase variable estimation errors satisfy linear perturbed dynamics whose trajectories also dominantly converge in an asymptotically exponentially dominated manner to small as desired neighborhoods of the origins of the reconstruction errors phase spaces. As a result, the disturbance vector components of $\boldsymbol{\varphi}(t) = [\varphi_x(t), \varphi_y(t), \varphi_F(t)]^T$ are closely estimated with an error bounded by a small as desired neighborhood of zero. As the location of the roots of the dominating characteristic polynomials are further pushed into the left half of the complex plane, all these tracking, or estimation, bounding neighborhoods become tighter around the origin.

4 Numerical Simulations

Numerical simulations were carried out in order to verify the efficiency of the proposed approach in terms of quick convergence of the tracking errors to a small neighborhood of zero, smooth transient responses and low control effort. In all the simulations the intention is to track a circular trajectory in a counter-clockwise sense in the plane x, y for the point O of the mobile base. The flat output F has additionally been designed as a sinusoidal function. In other words, the flat outputs are nominally specified as:

$$x^*(t) = R_1 \cos(\omega_1 t); \quad y^*(t) = R_1 \sin(\omega_1 t); \quad F^*(t) = A \cos(\omega_2 t) \tag{69}$$

where $R_1 = 15\,[m]$, $\omega_1 = 0.1\,[rad/s]$, $A = \pi/2\,[rad]$ and $\omega_1 = 0.06\,[rad/s]$. The time sampling used in all the simulations is $T = 0.001\,[s]$ and the values of the physical parameters of the underactuated wheeled mobile manipulator are depicted in Table 1. Furthermore, in order to demonstrate the exponential convergence of the desired trajectories, at the beginning of the simulations, the initial values of the flat outputs were selected with different values than the initial values of the nominal flat outputs. The observer gains, $\{\lambda_7^x, \ldots, \lambda_0^x\}$, $\{\lambda_7^y, \ldots, \lambda_0^y\}$ and $\{\lambda_8^F, \ldots, \lambda_0^F\}$, were selected by identifying, term by term, the coefficients of the polynomials given in expression (66) with those of a desired Hurwitz polynomial given by:

$$\hat{p}_x(s) = (s + r_1)^7; \quad \hat{p}_y(s) = (s + r_2)^7; \quad \hat{p}_F(s) = (s + r_3)^8 \tag{70}$$

with $r_1 = r_2 = r_3 = 15$. The controller gains, $\{k_n^x, \ldots, k_0^x\}$, $\{k_n^y, \ldots, k_0^y\}$ and $\{k_n^F, \ldots, k_0^F\}$, governing the dominant dynamics, were set by identifying, term by term, the coefficients of the polynomials given in expression (67) with the Hurwitz polynomials defined as follows:

$$p_x(s) = (s + b_1)^3; \quad p_y(s) = (s + b_2)^3; \quad p_F(s) = (s + b_3)^4 \tag{71}$$

with $b_1 = b_2 = b_3 = 0.5$. The coefficients of the desired Hurwitz polynomial for both the observer and the controller were chosen to meet a desirable

Two-Wheeled Mobile Manipulator	
Parameter	Value
d	$0.4\ [m]$
a	$0.2\ [m]$
R	$0.1\ [m]$
b	$0.2\ [m]$
m_0	$10\ [kg]$
I_0	$0.5\ [kg\,m^2]$
m_1	$1\ [kg]$
m_2	$1\ [kg]$
I_1	$0.1\ [kg\,m^2]$
I_2	$0.1\ [kg\,m^2]$
l_1	$0.3\ [m]$
l_2	$0.3\ [m]$
k_2	$1\ [N\,m/rad]$

Table 1 Parameters of the mobile manipulators used in the simulations

and convenient fast asymptotic and exponentially convergent dynamic to a neighborhood of zero for the error of estimation and the tracking error, respectively. Responses were obtained for two cases: (a) when the model parameters are perfectly known; and (b) when there are inaccuracies in the dynamic model of the order 5% in the mass of the links.

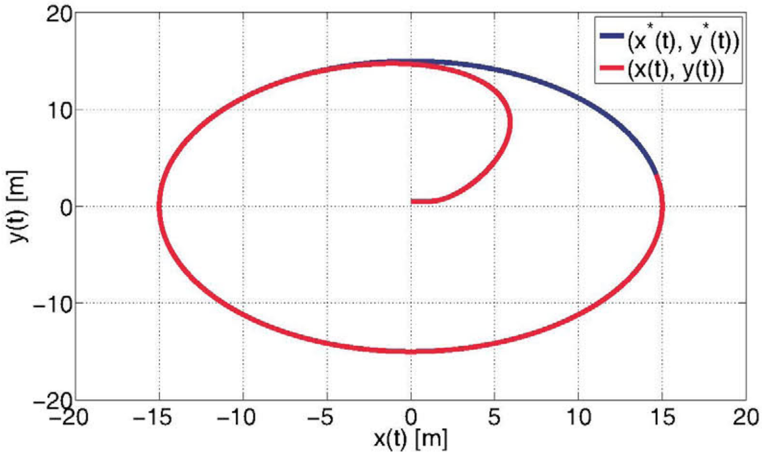

Fig. 2 Feedback controlled position coordinates of the point O of the Two-Wheeled Mobile Manipulator.

4.1 Tracking with accurate model

Figure 2 depicts the performance, in the (x, y) plane, of the proposed controller in the trajectory tracking problem for the mobile base position coordinates when the mobile manipulator is started significantly far away from the desired trajectory. Figure 3 illustrates the evolution of the flat outputs, Figure 4 shows the evolution of the orientation of the mobile base θ and the relative joint angles θ_1 and θ_2 of the manipulator arm and, finally, Figure 5 illustrates the applied external inputs τ_r, τ_l, τ_1 needed for the tracking. As will be observed, after a short period of time – needed for the errors to converge to a small neighborhood of zero – the underactuated robotic mobile manipulator follows the designed trajectory in an accurate manner.

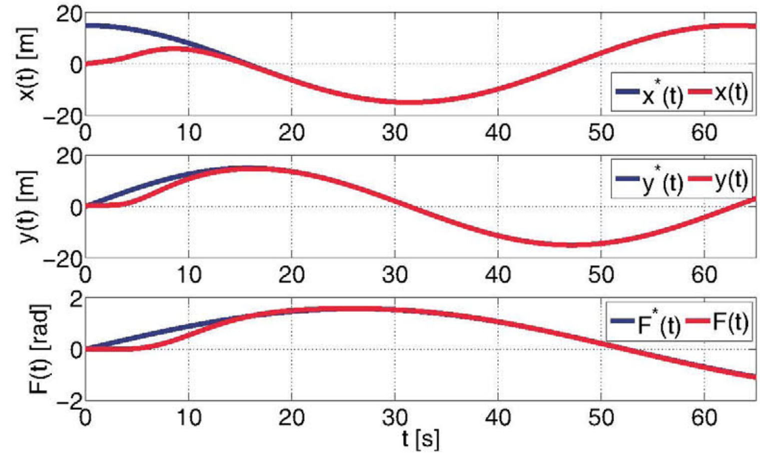

Fig. 3 Feedback controlled flat outputs of the Two-Wheeled Mobile Manipulator.

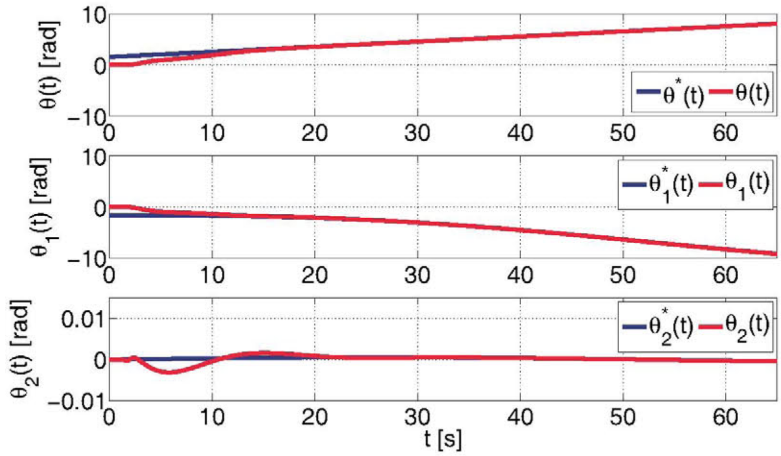

Fig. 4 Feedback controlled orientation of the mobile base, $\theta(t)$, and relative joint angles, θ_1 and θ_2, of the manipulator arm of the Two-Wheeled Mobile Manipulator.

4.2 Tracking with 5% error in some system parameters

In this section, we present some results concerning the robustness properties of the designed control scheme on the wheeled mobile manipulator under

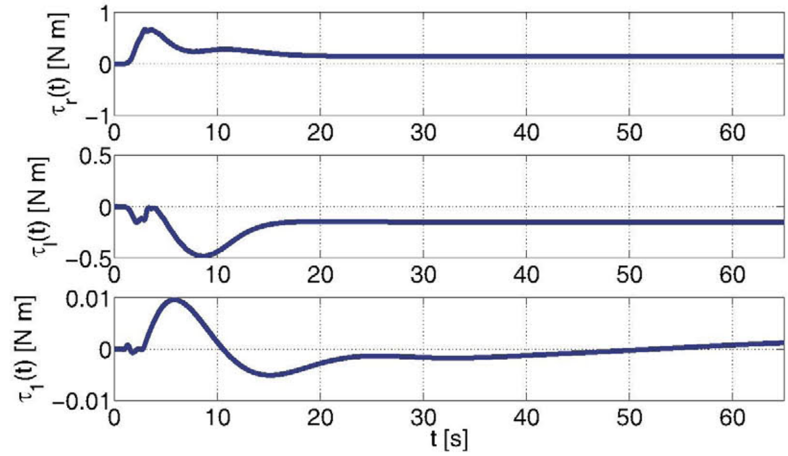

Fig. 5 Feedback controlled external inputs of the Two-Wheeled Mobile Manipulator.

large initial errors and model parametric uncertainties. We specifically conducted simulations in which the mobile manipulator is started significantly far away from the desired trajectory and the mass distribution of the manipulator is slightly modified, so that the center of mass is not precise located at the manipulator joint. In particular, errors in the model of the order of 5% on the mass of the links were assumed.

Figure 6 depicts the performance, in the (x, y) plane, of the proposed controller in the trajectory tracking problem for the mobile base position coordinates when the mobile manipulator is started significantly far away from the desired trajectory, Figure 7 illustrates the evolution of the flat outputs, Figure 8 shows the evolution of the orientation of the mobile base θ and the relative joint angles θ_1 and θ_2 of the manipulator arm, and Figure 9 illustrates the applied external inputs τ_r, τ_l, τ_1 needed for the tracking.

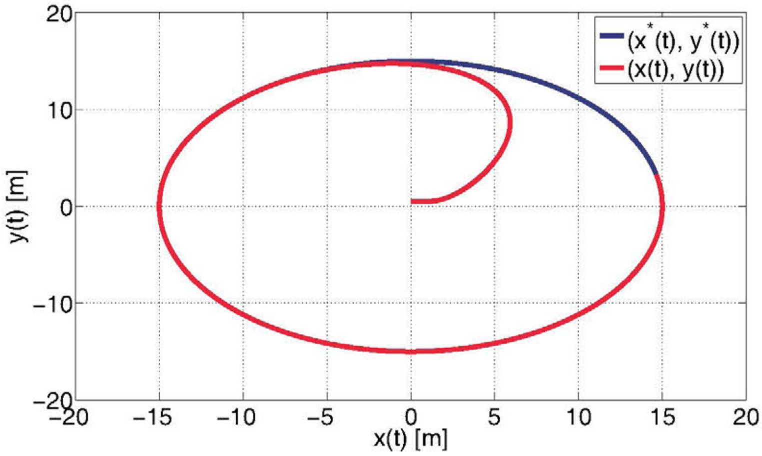

Fig. 6 Feedback controlled position coordinates of the point O of the Two-Wheeled Mobile Manipulator with 5% error in some system parameters.

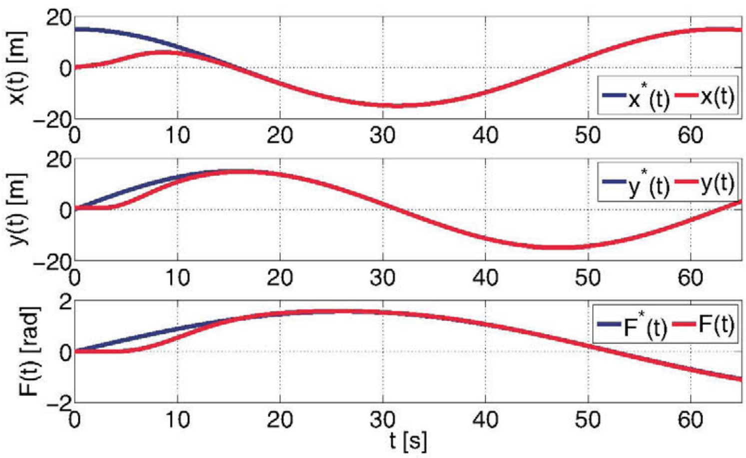

Fig. 7 Feedback controlled flat outputs of the Two-Wheeled Mobile Manipulator with 5% error system in some parameters.

As can be observed, in spite of the modeling errors, the proposed control algorithm corrects the motion of the wheeled mobile manipulator, guides the

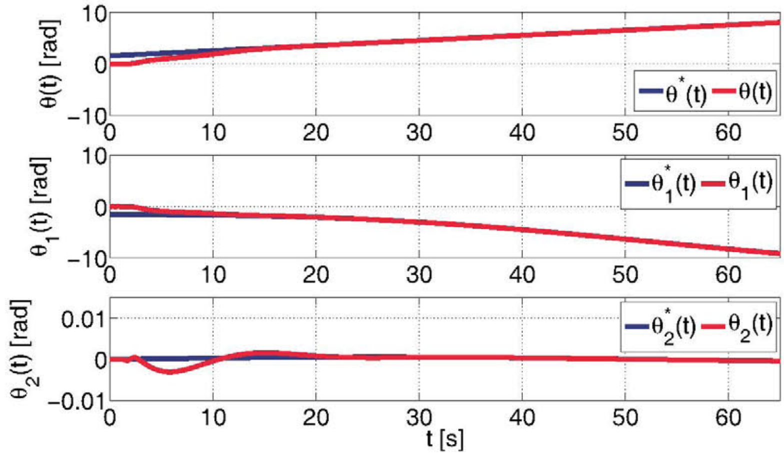

Fig. 8 Feedback controlled orientation of the mobile base, $\theta(t)$, and relative joint angles, θ_1 and θ_2, of the manipulator arm of the Two-Wheeled Mobile Manipulator with 5% error in some system parameters.

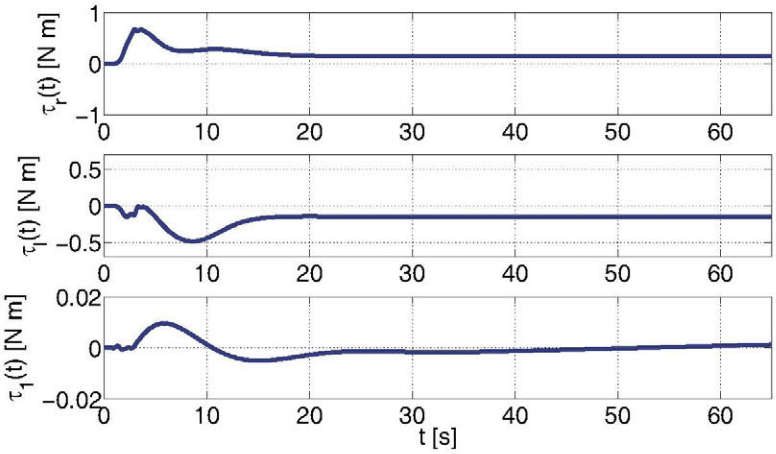

Fig. 9 Feedback controlled external inputs of the Two-Wheeled Mobile Manipulator with 5% error in some system parameters.

errors of the states to a small neighborhood of zero, and also compensates for the errors in the model parameters.

5 Conclusions

In this paper, we have described the design of an observer based robust linear output feedback controller for under-actuated planar mobile manipulators with a two-wheel differentially driven mobile base. The mobile base has nonholonomic constraints on the wheels and the manipulator is under-actuated of degree 1, i.e., the first joint is actuated while the second joint is underactuated but mounted with a torsion spring. The system is shown to be flat with flat outputs given by the horizontal and vertical positions, x and y, of the origin, O, of the mobile bases and the sum of the orientation of the mobile base θ and all the relative joint angles θ_i of the manipulator arm, i.e., $F = \theta + \sum_{i=1}^{2} \theta_i$. The input-to-flat-output dynamic system is modeled as a set of linear pure integration systems with a position dependent input gain matrix. The controller integration systems are influenced by additive absolutely bounded, yet observable, perturbation input signals, lumping all the unknown state dependent nonlinearities and the externally un-modeled inputs. Our basic assumption is that such perturbations can be locally approximated by an arbitrary representative of a fixed-degree family of time Taylor polynomials adopted as self-updating internal models in the GPI observer. The direct cancelation of the unknown perturbation inputs, via the linearizing feedback law, considerably simplifies the ultimate feedback controller design as regards classical linear feedback controllers with the derivative terms obtained from the GPI observers themselves. Digital computer simulations were provided, in which where the efficiency of the proposed control method was assessed with regard to large initial errors and substantial parametric uncertainties in the model. Future work will be devoted to verifying the application of the proposed control algorithms to reconfigurable stair-climbing mobility systems [34]-[36].

Finally, the GPI observer-based linear control of nonlinear systems is naturally fit for differentially flat systems, provided the flat output vector components are available for measurements. The fundamental restriction of unavailable flat outputs is yet to be fully explored. This topic, and other related limitations, need to be explored and resolved in the future, and we propose them as topics for further development. The resulting input-output description of the plant is radically simplified by assuming that important additive state-dependent terms, and unknown external perturbations, may be lumped in a uniformly absolutely bounded signal, treated as disturbances, or perturbation inputs.

6 Acknowledgements

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